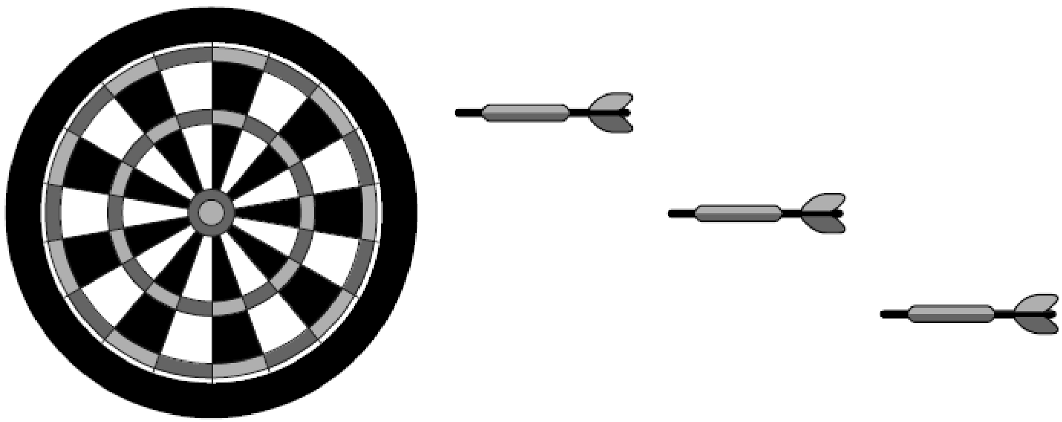
# AIHL_Paper_3_QP [55 marks]

**1.** [Maximum mark: 24]

**The following question examines the changes in darts players' scores using two statistical tests.**

In the sport of darts, players take turns throwing darts at a board in order to score points.



[Source: Panimoni, 2022. *Volume Target icon in flat style on color background. Darts game. Arrow in the center aim. Vector design element for you business projects.* [image online] Available at: https://www.gettyimages.co.uk/detail/illustration/volume-target-icon-in-flat-style-on-color-royalty-free-illustration/1044319572 [Accessed 21 February 2022]. Source adapted.]

A player's "three dart average" refers to the mean score achieved when throwing three darts.

Valia aimed to find out whether amateur darts players in her local area improved over a $12$-month period. An increase in their "three dart average" would indicate an improvement.

She selected a random sample of eight darts players and recorded their mean "three dart average" from Year $1$.

She then recorded their mean "three dart average" from Year $2$.

Valia's results were as follows:

**Table 1**

| Player | Year 1 mean | Year 2 mean |
|---|---|---|
| Justin | 68.1 | 70.1 |
| Fallon | 72.2 | 72.0 |
| Rob | 65.8 | 65.0 |
| Michael | 73.7 | 74.9 |
| Mensur | 69.8 | 68.0 |
| Deta | 64.5 | 68.0 |
| Meaghan | 69.2 | 70.6 |
| Peter | 64.9 | 92.9 |

Valia calculated the median, quartiles and inter-quartile range for each year. The results are shown in **Table 2**.

**Table 2**

| | Year 1 mean | Year 2 mean |
|---|---|---|
| **Lower quartile** | 65.35 | 68 |
| **Median** | 68.65 | $a$ |
| **Upper quartile** | 71 | $b$ |
| **Inter-quartile range** | 5.65 | $c$ |

(a) Determine the values of $a$, $b$ and $c$. [3]

(b) By comparing the results for both years summarized in **Table 2**, state one conclusion, in context, that Valia might be justified in making. [1]

Valia then decided to analyse the data from **Table 1** using a one-tailed paired $t$-test at the $10\%$ significance level to determine whether the players' averages have increased.

(c) State an assumption about the differences in means that is necessary in order for the test to be valid. [1]

(d.i) State the null and alternative hypotheses for the test. [2]

(d.ii) Find the $p$-value. [2]

(d.iii) State the conclusion of the test in context, justifying your answer. [2]

(e) State one way Valia could have reduced the chance of her making

(e.i) a Type I error. [2]

(e.ii) a Type II error. [2]

Valia was not sure the assumption made in part (c) was correct and hence thought the results obtained from her paired $t$-test may not be valid.

Following further research, Valia decided to use the Wilcoxon signed-rank test, which does not require the assumption she made in part (c).

For this test, the magnitudes of the differences between the Year $2$ and Year $1$ means are ranked from $1$ to $8$, with the ranks of the positive differences $(P)$ and the ranks of the negative differences $(N)$ separated in columns.

This is partially shown in the following table, which Valia constructs to perform the test.

| Year 1 mean | Year 2 mean | Difference | $P$ | $N$ |
|---|---|---|---|---|
| 68.1 | 70.1 | 2.0 | $A$ | |
| 72.2 | 72.0 | –0.2 | | 1 |
| 65.8 | 65.0 | –0.8 | | 2 |
| 73.7 | 74.9 | 1.2 | 3 | |
| 69.8 | 68.0 | –1.8 | | $B$ |
| 64.5 | 68.0 | 3.5 | $C$ | |
| 69.2 | 70.6 | 1.4 | $D$ | |
| 64.9 | 92.9 | 28 | 8 | |
| | | **Total** | $\sum P = 28$ | $\sum N$ |

(f) Determine the values of

(f.i) $A,\ B,\ C$ and $D$. [3]

(f.ii) $\Sigma N$. [1]

For this test:

- the Wilcoxon signed-rank test statistic is $T =$ the smaller value from a choice of $\Sigma P$ or $\Sigma N$.
- the null hypothesis is that the population's median for "three dart average" is the same in both years.

Valia chooses to carry out the test at the $5\%$ level of significance. From statistical tables, she determines that the critical region is $T \leq 5$.

(g.i) State the alternative hypothesis $\mathrm{H}_1$ for this test. [1]

(g.ii) Write down the value of the test statistic, $T$. [1]

(g.iii) Determine the conclusion of the test in context. [2]

(h) Suggest briefly how Valia could assess the reliability of her results for either test. [1]

**2.** [Maximum mark: 31]

**The following question explores a possible method of drawing phase portraits for non-linear coupled systems, taking a predator-prey model as a particular example.**

Gander Green wildlife park contains a population of Czech geese ($x$, measured in hundreds), and a population of gray foxes ($y$, measured in hundreds).

Research indicates that the population growth of both geese and foxes can be modelled by the following differential equations, in which $t$ is measured in years.

$$\left.\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= 2x - \frac{xy}{2}\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= -3y + xy\end{aligned}\right\}\text{ for } x,\ y \geq 0$$

(a) At a specific time, there are $500$ geese and $500$ foxes, represented here by the coordinate pair $(5,\ 5)$. At this time, determine the rate of change of

(a.i) geese. [2]

(a.ii) foxes. [1]

There are two equilibrium points for the populations: $\mathrm{A}(0,\ 0)$ and $\mathrm{B}(p,\ q)$.

(b.i) Explain why $\mathrm{A}$ is an equilibrium point. [1]

(b.ii) Find the value of $p$ and the value of $q$. [3]

At points close to $\mathrm{A}(0,\ 0)$, we can ignore the $xy$ terms, so that the system can be approximated by:

$$\left.\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= 2x\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= -3y\end{aligned}\right\}\text{ for } x,\ y \geq 0.$$

(c) By solving these two differential equations,

(c.i) find an expression for $x$ in terms of $t$. [4]

(c.ii) find an expression for $y$ in terms of $t$. [1]

(d.i) Using your answers from part (c), show that phase portrait trajectories close to $\mathrm{A}$ may be given by the equation $x^3y^2 = k$, where $k$ is a positive constant. [3]

(d.ii) Hence sketch, on a phase portrait, one possible trajectory for small values of $x$ and $y$. [3]

Now consider points $(x,\ y)$ close to $\mathrm{B}$ on the phase plane. These coordinates can be rewritten as $x = p + X$ and $y = q + Y$, where $p$ and $q$ are the values from part (b)(ii).

(e) By substituting into the original model, show that, for small values of $X$ and $Y$:

$\dot{X} \approx -\frac{3Y}{2}$. [3]

Similarly, it can be shown that $\dot{Y} \approx 4X$.

(f) Given that $\begin{pmatrix}\dot{X}\\ \dot{Y}\end{pmatrix} = \boldsymbol{M}\begin{pmatrix}X\\ Y\end{pmatrix}$, where $\boldsymbol{M}$ is a square matrix, write down $\boldsymbol{M}$. [1]

(g) By finding the eigenvalues of $\boldsymbol{M}$, describe the path of the trajectories close to point $\mathrm{B}$. [4]

(h) Hence sketch a complete set of trajectories in the phase plane for the original model, clearly indicating both equilibrium points.

[3]

(i) In this wildlife park, at a specific time, there are $500$ Czech geese and $500$ gray foxes.

Based on the values found in part (a), the park's wildlife keeper is worried and assumes that the geese will quickly die out. Suggest whether this assumption is supported by the model. Justify your answer. [2]