AIHL_Paper_2_QP [110 marks]

1. [Maximum mark: 13]

A shop uses the following model to estimate n, the number of smoothies sold per day, in terms of x, the price of a single smoothie in pesos.

$$n = \frac{40\,000}{x^2}$$

The maximum number of smoothies the shop can make in a day is 400.

(a) Find the maximum price they could charge per smoothie for the shop to sell 400 in one day. [2]

(b) On a day when the shop sells smoothies at 50 pesos each, use the model to find

(b.i) the number of smoothies sold. [1]

(b.ii) the total income from the smoothies sold. [1]

The cost of making each smoothie is 20 pesos. The profit per day (P) is the total income from the sale of smoothies that day minus the cost of making them.

(c.i) Show that, according to the model, $P = \frac{40\,000}{x} - \frac{800\,000}{x^2}$. [2]

(c.ii) Find $\frac{\mathrm{d}P}{\mathrm{d}x} = 0$. [3]

(c.iii) Find the value of x for which $\frac{\mathrm{d}P}{\mathrm{d}x} = 0$. [2]

(c.iv) Find the number of smoothies sold when the profit is maximized. [2]

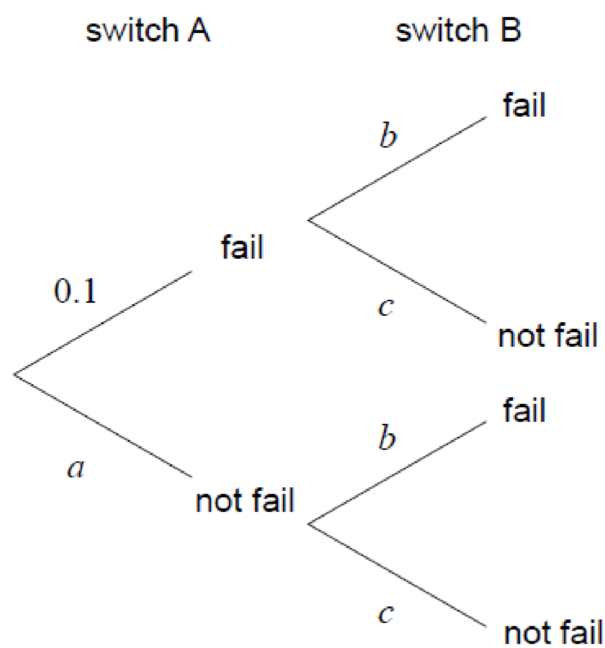
2. [Maximum mark: 12]

A type of generator will only function if a particular switch is working. The generator has a main switch, A, and a 'back up' switch, B.

The manufacturer claims the probability of switch A failing within one month of being fitted is 0.1 and the probability of the cheaper switch B failing within one month is 0.3. Whether or not a switch fails is independent of the state of the other switch.

If both switches fail, the generator needs to shut down to replace the switches. Both switches are replaced after a month of use (whether they have failed or not) or whenever the generator needs to be shut down.

The following tree diagram shows the probabilities of a switch failing within one month of them both being replaced, assuming the manufacturer's claim is correct.



(a) Write down the values of a, b and c. [2]

(b) Hence find the probability that the generator needs to shut down within one month of the switches being replaced. [1]

The owner of the generator is suspicious of the switch manufacturer's claims, so they look back through the past 200 occasions when the switches were replaced. The records show whether no switches, one switch or two switches had failed.

The data the owner collected are shown in the following table.

No switch fails	One switch fails	Two switches fail
118	72	10

(c) Perform a χ^2 goodness of fit test at the 5% significance level to test whether the manufacturer's claims are correct using the following hypotheses.

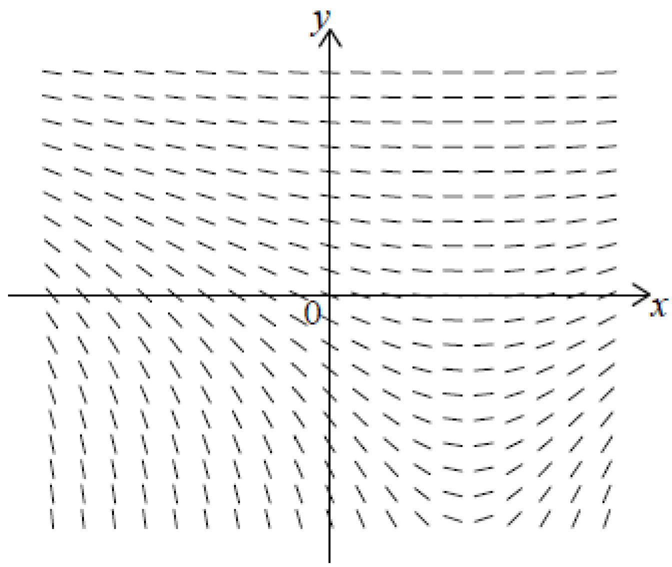
H_0 : The manufacturer's claims are correct.
H_1 : The manufacturer's claims are not both correct. [9]

3. [Maximum mark: 13]

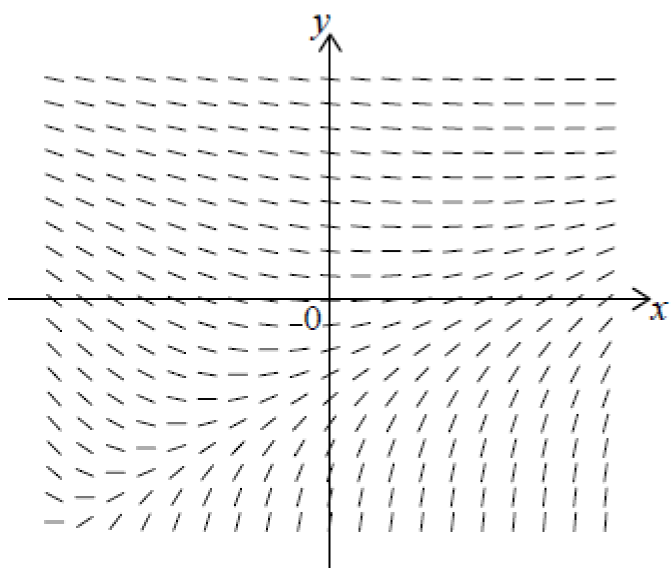
Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{\mathrm{e}^{2y}}$.

(a) Identify which of the following diagrams, **A**, **B** or **C**, represents the slope field for the differential equation. Give a reason for your answer.

A.

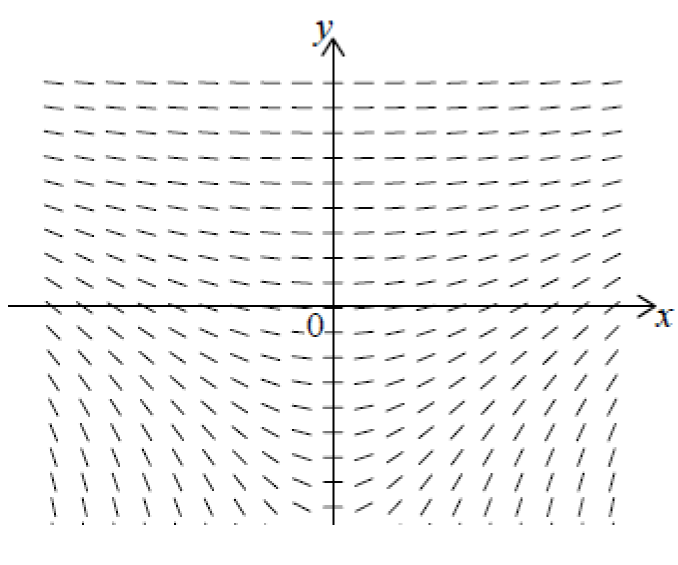


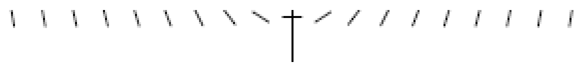
B.



[2]

C.





It is given that, for a particular solution, $x = 0$ and $y = 0$.

(b) Find an expression for y, in terms of x, for this solution. [7]

(c) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$, in terms of x, by differentiating your answer from part (b). [2]

(d) Hence verify that your answer to part (b) is a solution to $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{\mathrm{e}^{2y}}$. [2]

4. [Maximum mark: 13]

Taylor is playing a computer game in which they shoot at spaceships and battleships. The number of spaceships they hit per minute can be modelled by a Poisson distribution with mean 4.2. The number of battleships they hit per minute can be modelled by a Poisson distribution with a mean of 2.3. Any single hit occurs independently of all others.

(a) Find the probability Taylor hits

(a.i) at most 10 spaceships in 2 minutes. [2]

(a.ii) a total of more than 10 spaceships and battleships in one minute. [3]

Every spaceship that is hit earns Taylor 3 points and every battleship 5 points. Let T be the total points earned in one minute.

(b) Find

(b.i) $\mathrm{E}(T)$. [1]

(b.ii) $\mathrm{Var}(T)$. [2]

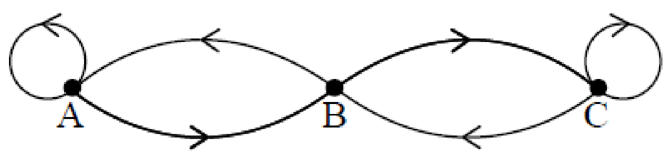
(c) State one reason why the distribution of T cannot be Poisson. [1]

Taylor intends to play the game for one hour.

(d) Use the central limit theorem to find the probability that Taylor's mean score per minute is greater than 25. [4]

5. [Maximum mark: 18]

(a) Write down the adjacency matrix for the directed graph shown below.



[2]

(b) Find the total number of walks of length 5 from A to B. [3]

A bird sits on one of three posts, labelled A, B and C, with B between A and C. When the bird moves, it will either fly to an adjacent post or return to the same post according to the following pattern.

- If it is on B, it will fly to A or C, each with a probability of 0.5.
- If it is on A or C, it will return to the same post with a probability of 0.5 or fly to B with a probability of 0.5.

The possible flights of the bird can be represented by the graph in part (a).

(c.i) Every possible sequence of 5 flights by the bird has the same probability of occurring. State this probability. [1]

(c.ii) Use your answer to part (b) to find the probability that if the bird was initially on post A, it will be on post B after 5 flights.

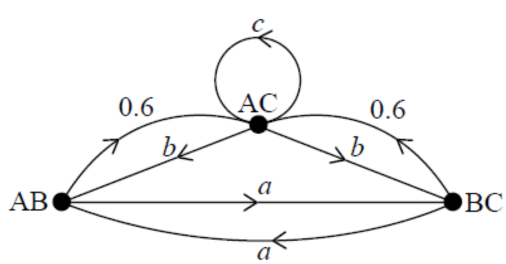
[2]

A second bird often joins the first on the posts. Their flights now follow the pattern given below.

- The birds will never sit on the same post.
- They will always fly from the posts at the same time.
- If they are on adjacent posts, the bird on post B will **always** fly to the vacant end post. The other bird will fly to post B with a probability of 0.4 or return to the same post with a probability of 0.6.
- If they are each on one of the end posts, they will fly to post B with a probability of 0.5 or return to the same post with a probability of 0.5. However, if they both try to fly to post B at the same time, they will see the other one doing so and both will immediately return to the post they were previously on.

The possible flights of the two birds can be represented by the following transition diagram, where the three vertices represent which posts are **occupied**.

diagram not to scale



(d) Write down the value of

(d.i) a. [1]

(d.ii) b. [1]

(d.iii) c. [1]

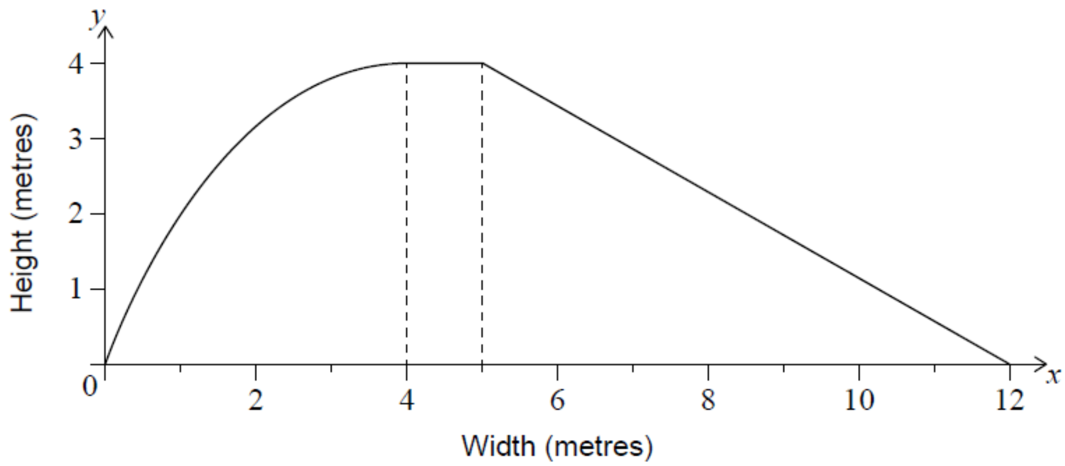
(e) Given that the birds are initially on posts A and B, find the probability they will be on posts B and C after 5 flights. [4]

The birds continue this pattern of flights for a long period.

(f) Given that the time between flights is always the same, find the post which is sat on least and the proportion of the time it is free. [3]

6. [Maximum mark: 15]

The following diagram shows a model of the side view of a water slide. All lengths are measured in metres.



The curved edge of the slide is modelled by

$f(x) = -\frac{1}{4}x^2 + 2x$ for $0 \leq x \leq 4$.

The remainder of the slide is modelled by

$$g(x) = \begin{cases} 4, & \text{for } 4 \leq \text{x} \leq 5 \\ \frac{48}{7} - \frac{4x}{7}, & \text{for } 5 \leq \text{x} \leq 12 \end{cases}$$

(a) Use the trapezoidal rule with an interval width of 1 to calculate the approximate area under the model of the slide in the interval $0 \leq x \leq 4$. [5]

(b) Find $\int \left(-\frac{1}{4}x^2 + 2x\right) \mathrm{d}x$. [3]

(c) Calculate the exact area under the entire model of the slide, for $0 \leq x \leq 12$. [4]

(d) Find the percentage error in the **total** area under the entire model of the slide when using the approximate value from part (a).

[3]

7. [Maximum mark: 14]

The k th triangle number, T_k, is defined as $T_k = \sum_{r=1}^{k} r$.

(a.i) Calculate the value of the fifth triangle number, T_5. [1]

(a.ii) Determine the formula for T_k in the form $ak^2 + bk$. [3]

(b.i) Find the value of $T_5 + T_4$. [1]

(b.ii) Find the simplest expression for $T_k + T_{k-1}$. [2]

A bag contains 15 red discs and 10 blue discs, all identical except for colour. Two discs are chosen at random from the bag without replacement.

(c) Calculate the probability that the two discs are different colours. [3]

A bag contains T_k red discs and T_{k-1} blue discs, all identical except for colour. Two discs are chosen at random from the bag without replacement.

(d) Show that the probability that the two discs are different colours is independent of k. [4]

8. [Maximum mark: 12]

The matrix A is defined by $A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$.

(a) Describe fully the geometrical transformation represented by A. [5]

Pentagon, P, which has an area of 7 cm^2, is transformed by A.

(b) Find the area of the image of P. [2]

The matrix B is defined by $B = \frac{1}{2}\begin{pmatrix} 3\sqrt{3} & 3 \\ -2 & 2\sqrt{3} \end{pmatrix}$.

B represents the combined effect of the transformation represented by a matrix X, followed by the transformation represented by A.

(c) Find the matrix X. [3]

(d) Describe fully the geometrical transformation represented by X. [2]