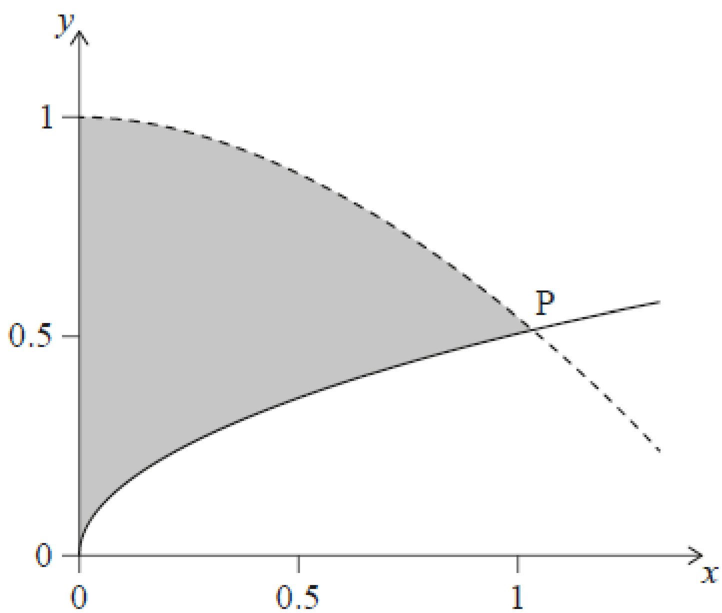
AIHL_Paper_1_QP [110 marks]

1. [Maximum mark: 9]

The following diagram shows parts of the curves of $y = \cos x$ and $y = \frac{\sqrt{x}}{2}$.

P is the point of intersection of the two curves.



(a) Use your graphic display calculator to find the coordinates of P. [2]

The shaded region is rotated 360° **about the y-axis** to form a volume of revolution V.

(b) Express V as the sum of two definite integrals. [5]

(c) Hence find the value of V. [2]

2. [Maximum mark: 6]

Consider the differential equation

$\left(x^2+1\right)\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{2y-2}$, for $x \geq 0, y \geq 1$,

where $y = 1$ when $x = 0$.

(a) Explain why Euler's method cannot be used to find an approximate value for y when $x = 0.1$. [1]

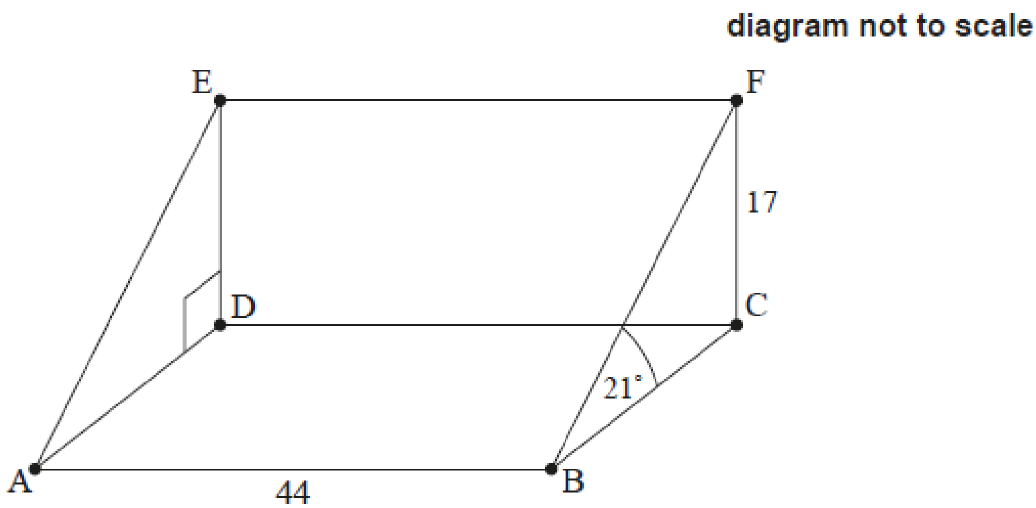
(b) By solving the differential equation, show that

$y = 1 + \sqrt{\frac{\ln\left(x^2+1\right)}{2}}$. [4]

(c) Hence deduce the value of y when $x = 0.1$. [1]

3. [Maximum mark: 5]

An artificial ski slope can be modelled as a triangular prism, as shown in the diagram. Rectangle ABCD is horizontal, and rectangle CDEF is vertical.



The maximum height of the ski slope, CF, is 17 metres and the steepest angle of the ski slope, $\mathrm{F\hat{B}C}$, is $21°$.

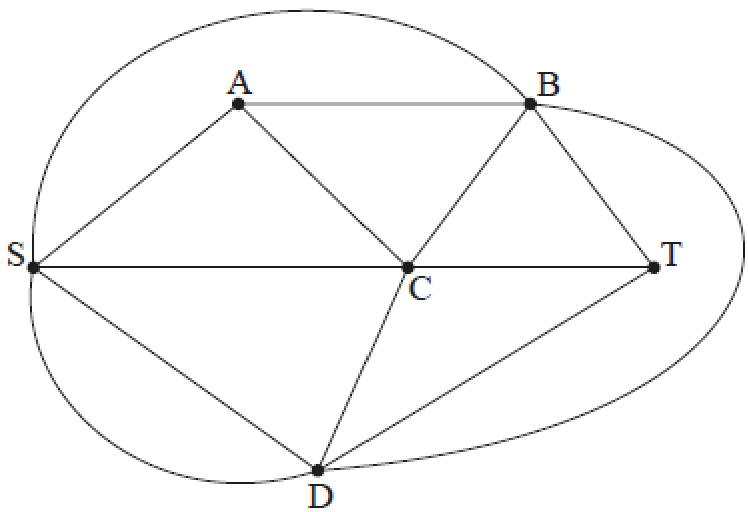
(a) Calculate the length of $[\mathrm{BF}]$. [2]

The width of the base of the ski slope, AB, is 44 metres. Mayumi skis in a straight line, starting from point E and finishing at the base of the ski slope.

(b) Find the value of the least steep angle that Mayumi can ski. [3]

4. [Maximum mark: 7]

In a competition, a contestant has to move through a maze to find treasure. A graph of the maze is shown below, where each edge represents a corridor in the maze. The contestant starts at S and the treasure is located at T.



(a) Complete the adjacency matrix, $\boldsymbol{M}$, for the graph.

	S	A	B	C	D	T
S	0	1	1	1	□	0
A	1	0	1	1	□	0
B	1	1	0	1	1	1
C	1	1	1	0	1	1
D	□	□	1	1	0	1
T	0	0	1	1	1	0

[2]

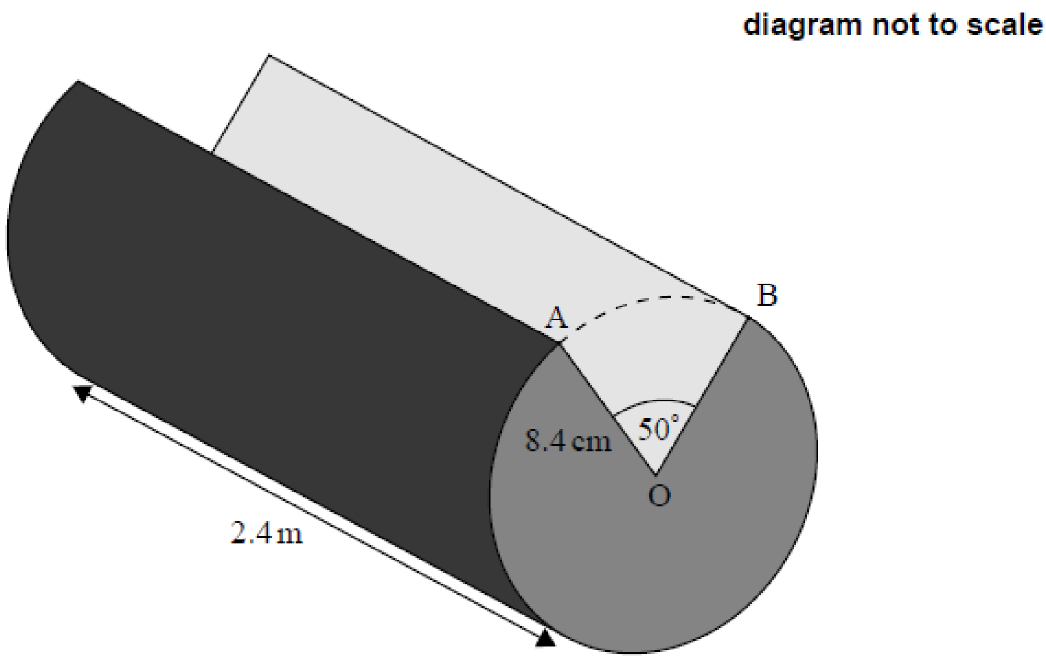
The competition rules state that the contestant can walk along a maximum of four corridors.

(b) Find the number of walks from S to T with a maximum of 4 edges. [4]

(c) Explain why the number of ways the contestant can reach the treasure is less than the answer to part (b). [1]

5. [Maximum mark: 5]

Helen is building a cabin using cylindrical logs of length 2.4 m and radius 8.4 cm. A wedge is cut from one log and the cross-section of this log is illustrated in the following diagram.



(a) Find 50° in radians. [1]

(b) Find the volume of this log. [4]

6. [Maximum mark: 5]

A particle, A, moves so that its velocity (ν ms^{-1}) at time t is given by $\nu = 2 \sin t, t \geq 0$.

The kinetic energy (E) of the particle A is measured in joules (J) and is given by $E = 5\nu^2$.

(a) Write down an expression for E as a function of time. [1]

(b) Hence find $\frac{\mathrm{d}E}{\mathrm{d}t}$. [2]

(c) Hence or otherwise find the first time at which the kinetic energy is changing at a rate of 5 J s^{-1}. [2]

7. [Maximum mark: 5]

A manager wishes to check the mean mass of flour put into bags in his factory. He randomly samples 10 bags and finds the mean mass is 1.478 kg and the standard deviation of the sample is 0.0196 kg.

(a) Find s_{n-1} for this sample. [2]

(b) Find a 95 % confidence interval for the population mean, giving your answer to 4 significant figures. [2]

(c) The bags are labelled as being 1.5 kg mass. Comment on this claim with reference to your answer in part (b). [1]

8. [Maximum mark: 6]

In a coffee shop, the time it takes to serve a customer can be modelled by a normal distribution with a mean of 1.5 minutes and a standard deviation of 0.4 minutes.

Two customers enter the shop together. They are served one at a time.

Find the probability that the total time taken to serve both customers will be less than 4 minutes.

Clearly state any assumptions you have made. [6]

9. [Maximum mark: 7]

Product research leads a company to believe that the revenue (R) made by selling its goods at a price (p) can be modelled by the equation.

$$R\left(p\right) = cpe^{dp},\ c, d \in \mathbb{R}$$

There are two competing models, A and B with different values for the parameters c and d.

Model A has $c = 3$, $d = -0.5$ and model B has $c = 2.5$, $d = -0.6$.

The company experiments by selling the goods at three different prices in three similar areas and the results are shown in the following table.

Area	Price (p)	Revenue (R)
1	1	1.5
2	2	1.8
3	3	1.5

The company will choose the model with the smallest value for the sum of square residuals.

Determine which model the company chose. [7]

10. [Maximum mark: 6]

The number of fish that can be caught in one hour from a particular lake can be modelled by a Poisson distribution.

The owner of the lake, Emily, states in her advertising that the average number of fish caught in an hour is three.

Tom, a keen fisherman, is not convinced and thinks it is less than three. He decides to set up the following test. Tom will fish for one hour and if he catches fewer than two fish he will reject Emily's claim.

(a) State a suitable null and alternative hypotheses for Tom's test. [1]

(b) Find the probability of a Type I error. [2]

(c) The average number of fish caught in an hour is actually 2.5.

Find the probability of a Type II error. [3]

11. [Maximum mark: 5]

Consider the following function, $f(x)$, defined on the domain of integers from 0 to 4 inclusive.

x	0	1	2	3	4
$f(x)$	3	1	0	4	2

(a) Solve $f(x) = 4$. [1]

(b) Solve $f(x) = x$. [1]

(c) Complete the following table.

x	0	1	2	3	4
$f^{-1}(x)$					

[3]

12. [Maximum mark: 6]

A biologist believes that there is a relationship between the possible population size of a group of birds (p thousand) and the population of a colony of wasps (w thousand). Based on her research she believes that the relationship is

$$w = p^3 - 4p^2 + 3p.$$

(a) When $w = 0$, find the possible values of p. [2]

(b) Determine the positive values of w for which there is only one positive value of p. [4]

13. [Maximum mark: 9]

A biologist uses a wire frame to count the number of worms in a $1\,\mathrm{m}^2$ section.

She models the number of worms found in each $1\,\mathrm{m}^2$ section as following a Poisson distribution with mean 1.2.

(a) Find the probability of observing exactly one worm in one $1\,\mathrm{m}^2$ section. [1]

(b) Find the probability of observing at least one worm in one $1\,\mathrm{m}^2$ section. [1]

The biologist looks at 5 independent $1\,\mathrm{m}^2$ sections.

(c) Find the probability of observing a total of five worms in 5 sections. [2]

(d) Find the probability of observing exactly one worm in all 5 sections. [2]

(e) Find the probability of observing at least one worm in exactly 3 of the 5 sections. [3]

14. [Maximum mark: 5]

A boat travels $8\ \text{km}$ on a bearing of $315°$ and then a further $6\ \text{km}$ on a bearing of $045°$.

Find the bearing on which the boat should travel to return directly to the starting point. [5]

15. [Maximum mark: 7]

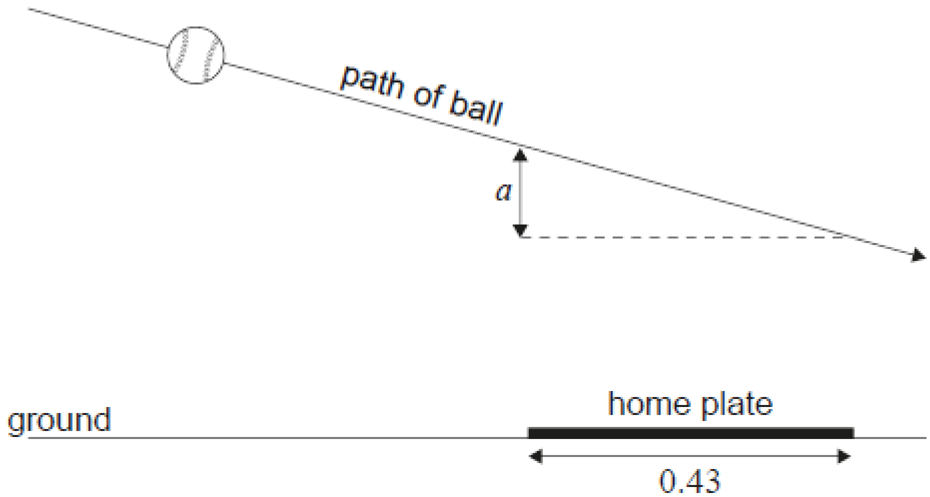
In a baseball game, Sakura is the batter standing beside home plate. The ball is thrown towards home plate along a path that can be modelled by the following function.

$$y = -0.045x + 2$$

In this model, x is the horizontal distance of the ball from the point the ball is thrown and y is the vertical height of the ball above the ground. Both measured in metres.

The outcome of the throw is called a strike if the height of the ball is between $0.53\,\text{m}$ and $1.24\,\text{m}$ at some point while it travels over home plate. The length of home plate is $0.43\,\text{m}$.

diagram not to scale



When the ball reaches the front of home plate, the height of the ball above the ground is $1.25\,\mathrm{m}$. The height of the ball changes by a metres as the ball travels over the length of home plate.

(a.i) Find the value of a. [2]

(a.ii) Justify why this throw is a strike. [2]

On the next throw, Sakura hits the ball towards a wall that is 5 metres high. The horizontal distance of the wall from the point where the ball was hit is 96 metres. The path of the ball after it is hit can be modelled by the function $h(d)$.

$h(d) = -0.01d^2 + 1.04d + 0.66$, for $h,\ d > 0$

In this model, h is the height of the ball above the ground and d is the horizontal distance of the ball from the point where it was hit. Both h and d are measured in metres.

(b) Determine whether the ball will go over the wall. Justify your answer. [3]

16. [Maximum mark: 5]

The function $f(x) = \ln\left(\frac{1}{x-2}\right)$ is defined for $x > 2,\ x \in \mathbb{R}$.

(a) Find an expression for $f^{-1}(x)$. You are not required to state a domain. [3]

(b) Solve $f(x) = f^{-1}(x)$. [2]

17. [Maximum mark: 5]

An electric circuit has two power sources. The voltage, V_1, provided by the first power source, at time t, is modelled by

$$V_1 = \mathrm{Re}\left(2\mathrm{e}^{3t\mathrm{i}}\right).$$

The voltage, V_2, provided by the second power source is modelled by

$$V_2 = \mathrm{Re}\left(5\mathrm{e}^{(3t+4)\mathrm{i}}\right).$$

The total voltage in the circuit, V_T, is given by

$$V_T = V_1 + V_2.$$

(a) Find an expression for V_T in the form $A\cos(Bt + C)$, where $A,\ B$ and C are real constants. [4]

(b) Hence write down the maximum voltage in the circuit. [1]

18. [Maximum mark: 7]

The wind chill index W is a measure of the temperature, in $^\circ\mathrm{C}$, felt when taking into account the effect of the wind.

When Frieda arrives at the top of a hill, the relationship between the wind chill index and the speed of the wind v in kilometres per hour $\left(\mathrm{km\,h^{-1}}\right)$ is given by the equation

$$W = 19.34 - 7.405v^{0.16}$$

(a) Find an expression for $\frac{\mathrm{d}W}{\mathrm{d}v}$. [2]

(b) When Frieda arrives at the top of a hill, the speed of the wind is 10 kilometres per hour and increasing at a rate of $5\,\mathrm{km\,h^{-1}\,minute^{-1}}$.

Find the rate of change of W at this time. [5]