

## AAHL\_Paper\_3\_QP [55 marks]

1. [Maximum mark: 25]

This question asks you to investigate some properties of the sequence of functions of the form

$$f_n(x) = \cos(n \arccos x), -1 \leq x \leq 1 \text{ and } n \in \mathbb{Z}^+.$$

**Important:** When sketching graphs in this question, you are **not** required to find the coordinates of any axes intercepts or the coordinates of any stationary points unless requested.

- (a) On the same set of axes, sketch the graphs of  $y = f_1(x)$  and  $y = f_3(x)$  for  $-1 \leq x \leq 1$ . [2]

For odd values of  $n > 2$ , use your graphic display calculator to systematically vary the value of  $n$ . Hence suggest an expression for odd values of  $n$  describing, in terms of  $n$ , the number of

- (b.i) local maximum points; [3]

- (b.ii) local minimum points; [1]

- (c) On a new set of axes, sketch the graphs of  $y = f_2(x)$  and  $y = f_4(x)$  for  $-1 \leq x \leq 1$ . [2]

For even values of  $n > 2$ , use your graphic display calculator to systematically vary the value of  $n$ . Hence suggest an expression for even values of  $n$  describing, in terms of  $n$ , the number of

- (d.i) local maximum points; [3]

- (d.ii) local minimum points. [1]

- (e) Solve the equation  $f_n'(x) = 0$  and hence show that the stationary points on the graph of  $y = f_n(x)$  occur at  $x = \cos \frac{k\pi}{n}$  where  $k \in \mathbb{Z}^+$  and  $0 < k < n$ . [4]

The sequence of functions,  $f_n(x)$ , defined above can be expressed as a sequence of polynomials of degree  $n$ .

- (f) Use an appropriate trigonometric identity to show that  $f_2(x) = 2x^2 - 1$ . [2]

Consider  $f_{n+1}(x) = \cos((n+1) \arccos x)$ .

- (g) Use an appropriate trigonometric identity to show that  $f_{n+1}(x) = \cos(n \arccos x) \cos(\arccos x) - \sin(n \arccos x) \sin(\arccos x)$ .

[2]

- (h.i) Hence show that  $f_{n+1}(x) + f_{n-1}(x) = 2x f_n(x), n \in \mathbb{Z}^+$ .

[3]

(h.ii) Hence express  $f_3(x)$  as a cubic polynomial.

[2]

2. [Maximum mark: 30]

A **Gaussian integer** is a complex number,  $z$ , such that  $z = a + bi$  where  $a, b \in \mathbb{Z}$ . In this question, you are asked to investigate certain divisibility properties of Gaussian integers.

Consider two Gaussian integers,  $\alpha = 3 + 4i$  and  $\beta = 1 - 2i$ , such that  $\gamma = \alpha\beta$  for some Gaussian integer  $\gamma$ .

(a) Find  $\gamma$ .

[2]

Now consider two Gaussian integers,  $\alpha = 3 + 4i$  and  $\gamma = 11 + 2i$ .

(b) Determine whether  $\frac{\gamma}{\alpha}$  is a Gaussian integer.

[3]

The norm of a complex number  $z$ , denoted by  $N(z)$ , is defined by  $N(z) = |z|^2$ . For example, if  $z = 2 + 3i$  then  $N(2 + 3i) = 2^2 + 3^2 = 13$ .

(c) On an Argand diagram, plot and label all Gaussian integers that have a norm less than 3.

[2]

(d) Given that  $\alpha = a + bi$  where  $a, b \in \mathbb{Z}$ , show that  $N(\alpha) = a^2 + b^2$ .

[1]

A **Gaussian prime** is a Gaussian integer,  $z$ , that **cannot** be expressed in the form  $z = \alpha\beta$  where  $\alpha, \beta$  are Gaussian integers with  $N(\alpha), N(\beta) > 1$ .

(e) By expressing the positive integer  $n = c^2 + d^2$  as a product of two Gaussian integers each of norm  $c^2 + d^2$ , show that  $n$  is not a Gaussian prime.

[3]

The positive integer 2 is a prime number, however it is not a Gaussian prime.

(f) Verify that 2 is not a Gaussian prime.

[2]

(g) Write down another prime number of the form  $c^2 + d^2$  that is not a Gaussian prime and express it as a product of two Gaussian integers.

[2]

Let  $\alpha, \beta$  be Gaussian integers.

(h) Show that  $N(\alpha\beta) = N(\alpha)N(\beta)$ .

[6]

The result from part (h) provides a way of determining whether a Gaussian integer is a Gaussian prime.

(i) Hence show that  $1 + 4i$  is a Gaussian prime. [3]

(j) Use proof by contradiction to prove that a prime number,  $p$ , that is not of the form  $a^2 + b^2$  is a Gaussian prime. [6]