AAHL_Paper_3_QP [55 marks]

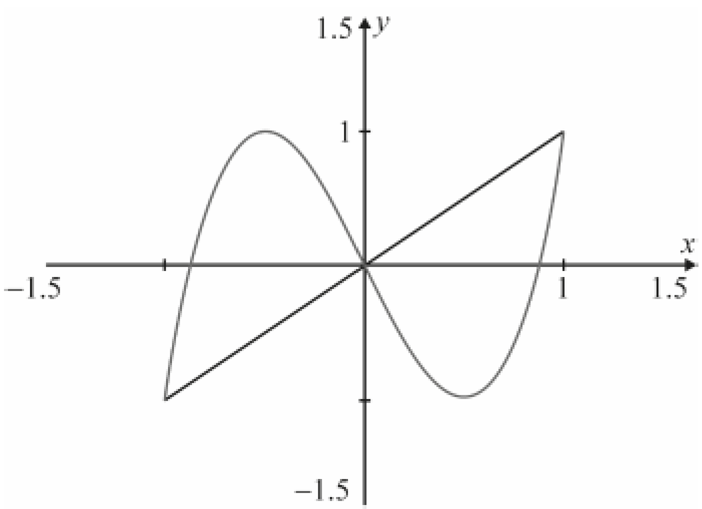
1. [Maximum mark: 25]

(a) [2]

Markscheme

correct graph of $y = f_1(x)$ ***A1***

correct graph of $y = f_3(x)$ ***A1***



[2 marks]

(b.i) [3]

Markscheme

graphical or tabular evidence that n has been systematically varied ***M1***

eg n = 3, 1 local maximum point and 1 local minimum point

n = 5, 2 local maximum points and 2 local minimum points

n = 7, 3 local maximum points and 3 local minimum points ***(A1)***

$\frac{n-1}{2}$ local maximum points ***A1***

[3 marks]

(b.ii) [1]

Markscheme

$\frac{n-1}{2}$ local minimum points ***A1***

Note: Allow follow through from an incorrect local maximum formula expression.

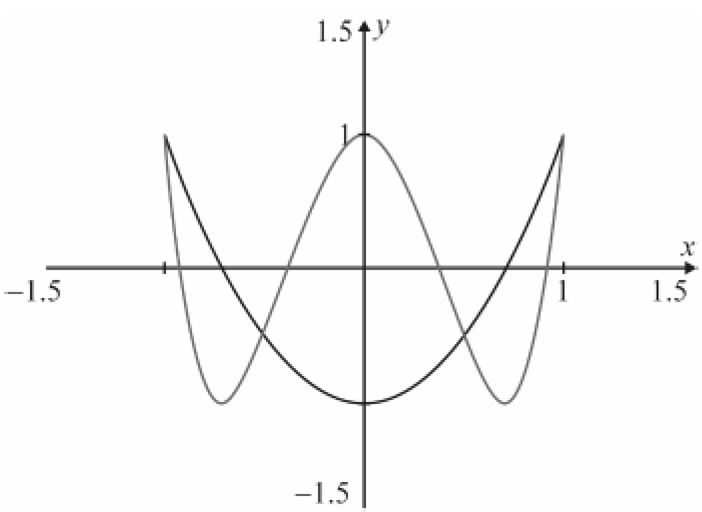
[1 mark]

(c) [2]

Markscheme

correct graph of $y = f_2(x)$ ***A1***

correct graph of $y = f_4(x)$ ***A1***



[2 marks]

(d.i) [3]

Markscheme

graphical or tabular evidence that n has been systematically varied ***M1***

eg n = 2, 0 local maximum point and 1 local minimum point

n = 4, 1 local maximum points and 2 local minimum points

n = 6, 2 local maximum points and 3 local minimum points ***(A1)***

$\frac{n-2}{2}$ local maximum points ***A1***

[3 marks]

(d.ii) [1]

Markscheme

$\frac{n}{2}$ local minimum points ***A1***

[1 mark]

(e) [4]

Markscheme

$$f_n(x) = \cos\left(n \arccos\left(x\right)\right)$$

$$f_n{}'(x) = \frac{n \sin(n \arccos(x))}{\sqrt{1-x^2}}$$ ***M1A1***

Note: Award ***M1*** for attempting to use the chain rule.

$$f_n{}'(x) = 0 \Rightarrow n \sin\left(n \arccos\left(x\right)\right) = 0$$ ***M1***

$$n \arccos\left(x\right) = k\pi \quad \left(k \in \mathbb{Z}^+\right)$$ ***A1***

leading to

$x = \cos\frac{k\pi}{n}$ ($k \in \mathbb{Z}^+$ and $0 < k < n$) ***AG***

[4 marks]

(f) [2]

Markscheme

$f_2(x) = \cos(2\arccos x)$

$= 2(\cos(\arccos x))^2 - 1$ ***M1***

stating that $(\cos(\arccos x)) = x$ ***A1***

so $f_2(x) = 2x^2 - 1$ ***AG***

[2 marks]

(g) [2]

Markscheme

$f_{n+1}(x) = \cos((n+1)\arccos x)$

$= \cos(n\arccos x + \arccos x)$ ***A1***

use of $\cos(A+B) = \cos A\cos B - \sin A\sin B$ leading to ***M1***

$= \cos(n\arccos x)\cos(\arccos x) - \sin(n\arccos x)\sin(\arccos x)$ ***AG***

[2 marks]

(h.i) [3]

Markscheme

$f_{n-1}(x) = \cos((n-1)\arccos x)$ ***A1***

$= \cos(n \arccos x)\cos(\arccos x) + \sin(n \arccos x)\sin(\arccos x)$ ***M1***

$f_{n+1}(x) + f_{n-1}(x) = 2\cos(n \arccos x)\cos(\arccos x)$ ***A1***

$= 2x f_n(x)$ ***AG***

[3 marks]

(h.ii) [2]

Markscheme

$f_3(x) = 2x f_2(x) - f_1(x)$ ***(M1)***

$= 2x\left(2x^2 - 1\right) - x$

$= 4x^3 - 3x$ ***A1***

[2 marks]

2. [Maximum mark: 30]

(a) [2]

Markscheme

* This sample question was produced by experienced DP mathematics senior examiners to aid teachers in preparing for external assessment in the

new MAA course. There may be minor differences in formatting compared to formal exam papers.

$(3+4\mathrm{i})(1-2\mathrm{i})=11-2\mathrm{i}$ ***(M1)A1***

[2 marks]

(b) [3]

Markscheme

$\frac{\gamma}{\alpha}=\frac{41}{25}-\frac{38}{25}\mathrm{i}$ ***(M1)A1***

(Since $\mathrm{Re}\frac{\gamma}{\alpha}\left(=\frac{41}{25}\right)$ and/or $\mathrm{Im}\frac{\gamma}{\alpha}\left(=-\frac{38}{25}\right)$ are not integers)

$\frac{\gamma}{\alpha}$ is not a Gaussian integer ***R1***

Note: Award ***R1*** for correct conclusion from their answer.

[3 marks]

(c) [2]

Markscheme

$\pm 1,\ \pm\mathrm{i},\ 0$ plotted and labelled ***A1***

$1\pm\mathrm{i},\ -1\pm\mathrm{i}$ plotted and labelled ***A1***

Note: Award ***A1A0*** if extra points to the above are plotted and labelled.

[2 marks]

(d) [1]

Markscheme

$|z| = \sqrt{a^2 + b^2}$ (and as $N(z) = |z|^2$) ***A1***

then $N(\alpha) = a^2 + b^2$ ***AG***

[1 mark]

(e) [3]

Markscheme

$c^2 + d^2 = (c + d\mathrm{i})(c - d\mathrm{i})$ ***A1***

and $N(c + d\mathrm{i}) = N(c - d\mathrm{i}) = c^2 + d^2$ ***R1***

$N(c + d\mathrm{i}),\ N(c - d\mathrm{i}) > 1$ (since $c,\ d$ are positive) ***R1***

so $c^2 + d^2$ is not a Gaussian prime, by definition ***AG***

[3 marks]

(f) [2]

Markscheme

$2\left(=1^2+1^2\right)=(1+\mathrm{i})(1-\mathrm{i})$ ***(A1)***

$N(1+\mathrm{i})=N(1-\mathrm{i})=2$ ***A1***

so 2 is not a Gaussian prime ***AG***

[2 marks]

(g) [2]

Markscheme

For example, $5\left(=1^2+2^2\right)=(1+2\mathrm{i})(1-2\mathrm{i})$ ***(M1)A1***

[2 marks]

(h) [6]

Markscheme

METHOD 1

Let $\alpha=m+n\mathrm{i}$ and $\beta=p+q\mathrm{i}$

LHS:

$\alpha\beta=(mp-nq)+(mq+np)\mathrm{i}$ ***M1***

$N(\alpha\beta)=(mp-nq)^2+(mq+np)^2$ ***A1***

$(mp)^2-2mnpq+(nq)^2+(mq)^2+2mnpq+(np)^2$ ***A1***

$(mp)^2+(nq)^2+(mq)^2+(np)^2$ ***A1***

RHS:

$N(\alpha)N(\beta) = \left(m^2 + n^2\right)\left(p^2 + q^2\right)$ ***M1***

$(mp)^2 + (mq)^2 + (np)^2 + (nq)^2$ ***A1***

LHS = RHS and so $N(\alpha\beta) = N(\alpha)N(\beta)$ ***AG***

METHOD 2

Let $\alpha = m + n\text{i}$ and $\beta = p + q\text{i}$

LHS

$N(\alpha\beta) = \left(m^2 + n^2\right)\left(p^2 + q^2\right)$ ***M1***

$= (m + n\text{i})(m - n\text{i})(p + q\text{i})(p - q\text{i})$ ***A1***

$= (m + n\text{i})(p + q\text{i})(m - n\text{i})(p - q\text{i})$

$= ((mp - nq) + (mq + np)\text{i})((mp - nq) - (mq + np)\text{i})$ ***M1A1***

$= (mp - nq)^2 + (mq + np)^2$ ***A1***

$N = ((mp - nq) + (mq + np)\text{i})$ ***A1***

$= N(\alpha)N(\beta)$ (= RHS) ***AG***

[6 marks]

(i) [3]

Markscheme

$N(1 + 4\text{i}) = 17$ which is a prime (in $\mathbb{Z}$) ***R1*** if $1+4\mathrm{i}=\alpha\beta$ then $17=N(\alpha\beta)=N(\alpha)N(\beta)$ ***R1***

we cannot have $N(\alpha),\ N(\beta)>1$ ***R1***

Note: Award ***R1*** for stating that $1+4\mathrm{i}$ is not the product of Gaussian integers of smaller norm because no such norms divide 17

so $1+4\mathrm{i}$ is a Gaussian prime ***AG***

[3 marks]

(j) [6]

Markscheme

Assume p is not a Gaussian prime

$\Rightarrow p=\alpha\beta$ where $\alpha,\ \beta$ are Gaussian integers and $N(\alpha),\ N(\beta)>1$ ***M1***

$\Rightarrow N(p)=N(\alpha)N(\beta)$ ***M1***

$p^2=N(\alpha)N(\beta)$ ***A1***

It cannot be $N(\alpha)=1,\ N(\beta)=p^2$ from definition of Gaussian prime ***R1***

hence $N(\alpha)=p,\ N(\beta)=p$ ***R1***

If $\alpha=a+b\mathrm{i}$ then $N(\alpha)=a^2+b^2=p$ which is a contradiction ***R1*** hence a prime number, p, that is not of the form $a^2 + b^2$ is a Gaussian prime ***AG***

[6 marks]