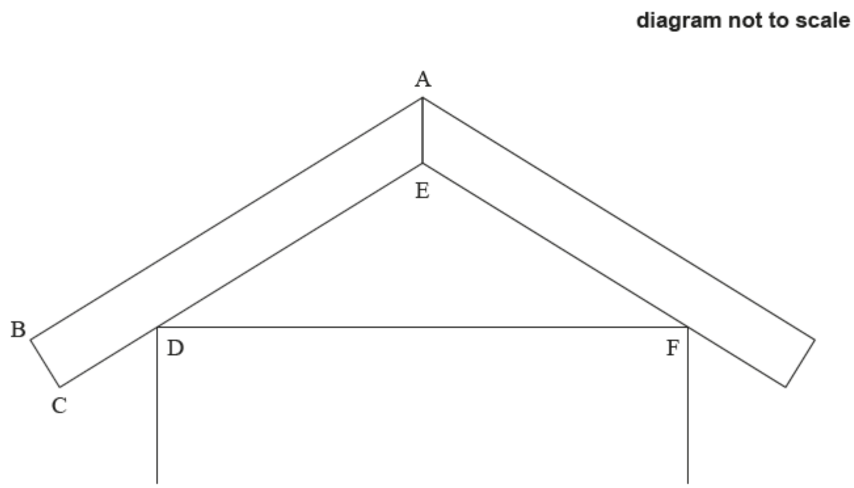
# AAHL_Paper_2_QP_(2 hrs) [110 marks]

**1.** [Maximum mark: 8]

The following diagram shows the cross section of the roof of a house. The cross section is symmetrical about the vertical line through points $\mathrm{A}$ and $\mathrm{E}$.



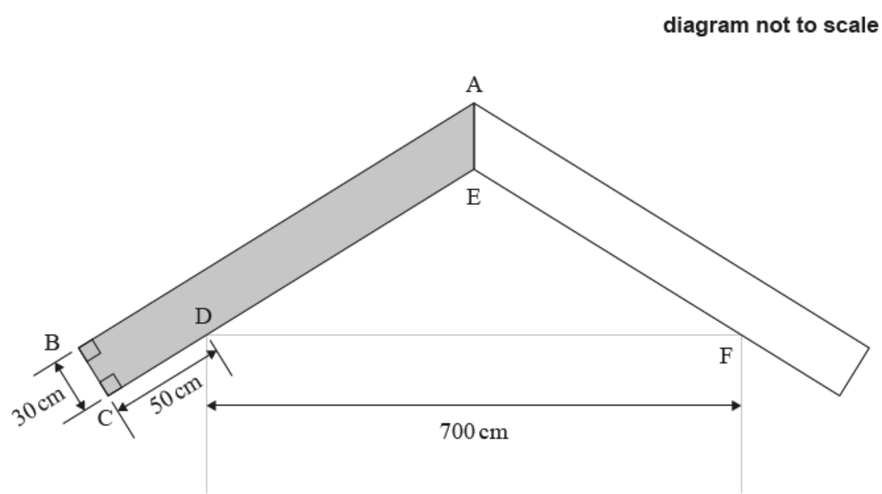

The gradient of $[\mathrm{BA}]$ is $\frac{7}{12}$.

(a) Find the size of $\mathrm{B\hat{A}E}$, expressing your answer in degrees. [3]

A builder requires the lengths of the sides $[\mathrm{BA}]$ and $[\mathrm{CE}]$.

The builder has the following measurements:

$\mathrm{A\hat{B}C} = \mathrm{B\hat{C}E} = 90^{\circ}$, $\mathrm{DC} = 50\ \mathrm{cm}$, $\mathrm{BC} = 30\ \mathrm{cm}$, and $\mathrm{DF} = 700\ \mathrm{cm}$.




(b) Find

(b.i) $\mathrm{CE}$ ;

[3]

(b.ii) $\mathrm{BA}$. [2]

**2.** [Maximum mark: 5]

Consider the function $h(x) = \log_{10}\left(4x^2 - rx + r - 1\right)$, where $x \in \mathbb{R}$.

Find the possible values of $r$. [5]

**3.** [Maximum mark: 8]

A continuous random variable $X$ has probability density function

$$f(x) = \begin{cases} \frac{1}{5} & 0 \le x < 2 \\ -\frac{x}{30} + \frac{4}{15} & 2 \le x \le 8 \\ 0 & \text{otherwise.} \end{cases}$$

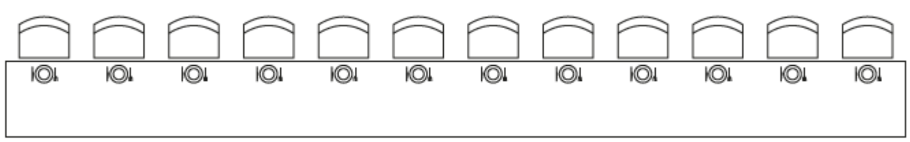
(a) Find $\mathrm{E}(X)$. [3]

(b) Given that $\mathrm{E}(c - 2X) = 0$, where $c$ is a constant, determine the value of $c$. [2]

(c) Find the median of $X$. [3]

**4.** [Maximum mark: 6]

A self-service sushi restaurant has a row of $12$ available seats, as shown in the following diagram.



Anvi, Vanya and Parita decide to go to the restaurant for lunch.

(a) Find the number of possible ways that they can be seated in this row, if they decide **not** to sit together as a group of $3$. [3]

The next day, Anvi, Vanya and Parita are joined by $3$ additional people in the same restaurant, and they sit in the same row of $12$ available seats. Anvi, Vanya and Parita now decide to sit next to each other as a group of $3$.

(b) Find the number of possible ways that these $6$ people can be seated. [3]

**5.** [Maximum mark: 7]

The sum of the first $n$ terms of a geometric sequence is given by $S_n = \sum_{r=0}^{n-1} 5\left(\log_2 c\right)^r$.

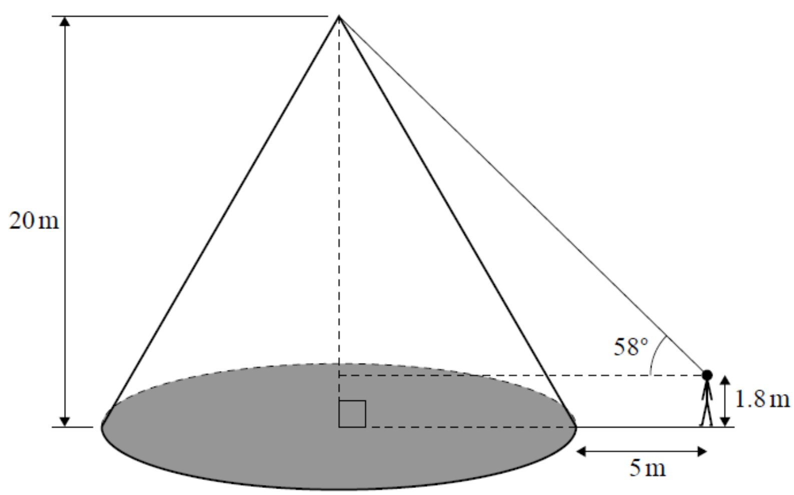
(a) Given that $S_n$ converges, find the range of possible values of $c$. [3]

(b) In the case where $c = 1.5$, find the least value of $n$ such that $|S_\infty - S_n| < 0.1$. [4]

**6.** [Maximum mark: 5]

A monument is in the shape of a right cone with a vertical height of $20$ metres. Oliver stands $5$ metres from the base of the monument. His eye level is $1.8$ metres above the ground and the angle of elevation from Oliver's eye level to the vertex of the cone is $58^\circ$, as shown on the following diagram.

**diagram not to scale**



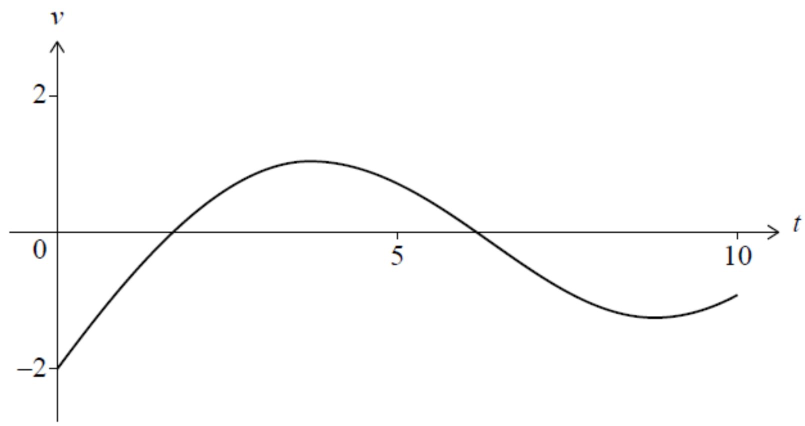

(a) Find the radius of the base of the cone. [3]

(b) Find the volume of the monument. [2]

**7.** [Maximum mark: 6]

A particle moves in a straight line such that it passes through a fixed point O at time $t = 0$, where $t$ represents time measured in seconds after passing O. For $0 \leq t \leq 10$ its velocity, $v$ metres per second, is given by $v = 2\sin(0.5t) + 0.3t - 2$.

The graph of $v$ is shown in the following diagram.




(a) Find the smallest value of $t$ when the particle changes direction. [2]

The displacement of the particle is measured in metres from $\mathrm{O}$.

(b) Find the range of values of $t$ for which the velocity is positive. [2]

(c) Find the displacement of the particle relative to $\mathrm{O}$ when $t = 10$. [2]

**8.** [Maximum mark: 8]

Consider $\lim_{x \to 0} \frac{\arctan(\cos x) - k}{x^2}$, where $k \in \mathbb{R}$.

(a) Show that a finite limit only exists for $k = \frac{\pi}{4}$. [2]

(b) Using l'Hôpital's rule, show algebraically that the value of the limit is $-\frac{1}{4}$. [6]

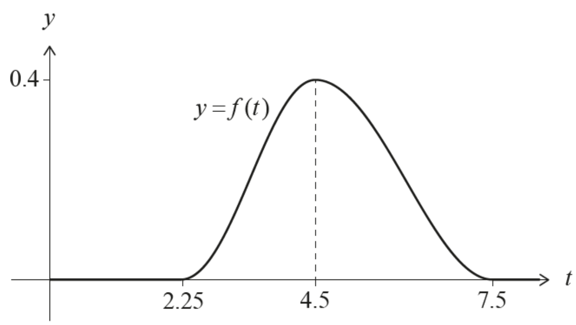
**9.** [Maximum mark: 19]

In a marathon race, the random variable $T$ represents the time, in hours, taken for a runner to complete the race. No runner completes the race in less than $2.25$ hours, and no runner completes it in more than $7.5$ hours.

The probability density function for $T$ is modelled by $f$, defined by

$$f(t) = \begin{cases} \frac{4}{21}\left(1 - \cos\left(\frac{4\pi}{9}(t - 2.25)\right)\right), & 2.25 \leq t < 4.5 \\ \frac{4}{21}\left(1 + \cos\left(\frac{\pi}{3}(t - 4.5)\right)\right), & 4.5 \leq t < 7.5 \\ 0, & \text{otherwise.} \end{cases}$$

The graph of $f$ has a maximum point at $t = 4.5$ as shown in the following diagram:




(a.i) Find the value of $\int_{2.25}^{4.5} f(t) \, \mathrm{d}t$. [1]

(a.ii) Write down the mode of $T$. [1]

(a.iii) Determine which is greater, the mode of $T$ or the median of $T$, justifying your answer. [2]

The runners who finish the race in $3.5$ hours or less are considered to be fast runners.

(b) Find the probability that a runner chosen at random is a fast runner. [2]

(c) Find the probability that a fast runner chosen at random finishes the race in $3$ hours or less. [3]

(d) Find the lower quartile of $T$. [3]

Each runner's time is converted to a score which is calculated as $a - bt$, where $t$ represents their time in hours, and $a,\ b > 0$.

Consider the random variable $P$ which represents the score of a runner. It is given that $\mathrm{E}(P) = 100$ and the maximum possible score is $150$.

(e) Use $\mathrm{E}(T) = 4.723$ to determine the value of $a$ and the value of $b$, giving your answers to the nearest integer. [5]

(f) Given also that $\mathrm{Var}\ (T) = 0.906$, find $\mathrm{Var}\ (P)$. [2]

**10.** [Maximum mark: 17]

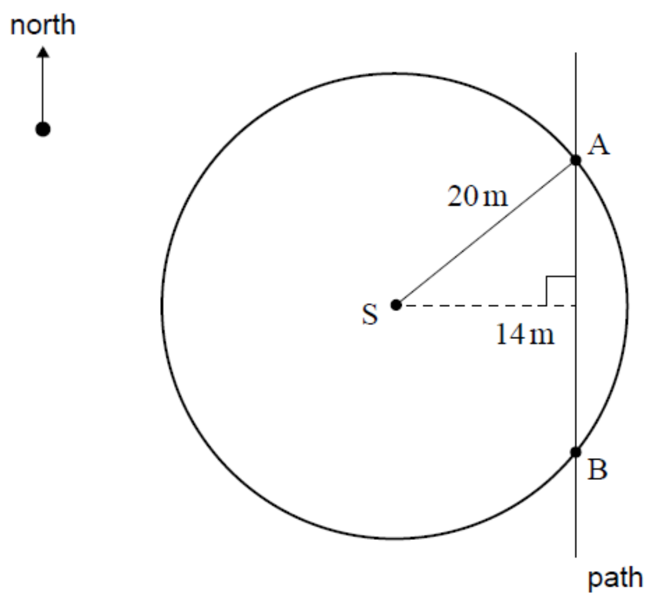
A rotating sprinkler is at a fixed point $\mathrm{S}$.

It waters all points inside and on a circle of radius $20$ metres.

Point $\mathrm{S}$ is $14$ metres from the edge of a path which runs in a north-south direction.

The edge of the path intersects the circle at points $\mathrm{A}$ and $\mathrm{B}$.

This information is shown in the following diagram.




(a) Show that $\text{AB} = 28.57$, correct to four significant figures. [3]

The sprinkler rotates at a constant rate of one revolution every $16$ seconds.

(b) Show that the sprinkler rotates through an angle of $\frac{\pi}{8}$ radians in one second. [1]

Let $T$ seconds be the time that $[\text{AB}]$ is watered in each revolution.

(c) Find the value of $T$. [4]

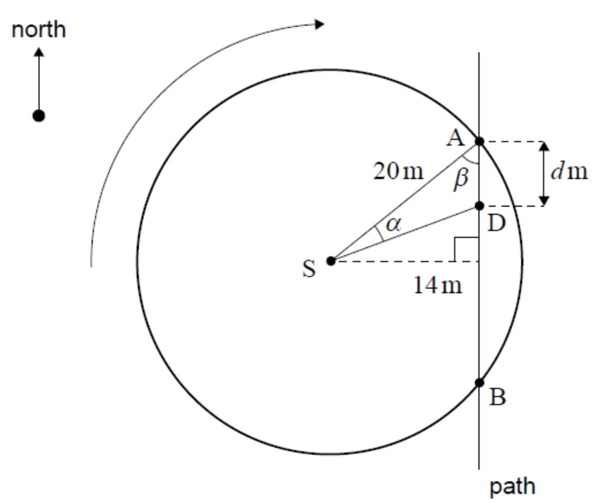
Consider one clockwise revolution of the sprinkler.

At $t = 0$, the water crosses the edge of the path at $\text{A}$.

At time $t$ seconds, the water crosses the edge of the path at a movable point $\text{D}$ which is a distance $d$ metres south of point $\text{A}$.

Let $\alpha = \text{A}\widehat{\text{S}}\text{D}$ and $\beta = \text{S}\widehat{\text{A}}\text{B}$, where $\alpha,\ \beta$ are measured in radians.

This information is shown in the following diagram.



(d) Write down an expression for $\alpha$ in terms of $t$. [1]

It is known that $\beta = 0.7754$ radians, correct to four significant figures.

(e) By using the sine rule in $\Delta\mathrm{ASD}$, show that the distance, $d$, at time $t$, can be modelled by

$$d(t) = \frac{20\sin\left(\frac{\pi t}{8}\right)}{\sin\left(2.37 - \frac{\pi t}{8}\right)}.$$

[3]

A turtle walks south along the edge of the path.

At time $t$ seconds, the turtle's distance, $g$ metres south of $\mathrm{A}$, can be modelled by

$g(t) = 0.05t^2 + 1.1t + 18$, where $t \geq 0$.

(f) At $t = 0$, state how far south the turtle is from $\mathrm{A}$. [1]

Let $w$ represent the distance between the turtle and point $\mathrm{D}$ at time $t$ seconds.

(g.i) Use the expressions for $g(t)$ and $d(t)$ to write down an expression for $w$ in terms of $t$. [1]

(g.ii) Hence find when and where on the path the water first reaches the turtle. [3]

**11.** [Maximum mark: 21]

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} - y\,\mathrm{cosec}\,2x = \sqrt{\tan x}$, where $0 < x < \frac{\pi}{2}$ and $y = \frac{\pi}{4}$ at $x = \frac{\pi}{4}$.

(a) Use Euler's method with step length $\frac{\pi}{12}$ to find an approximate value of $y$ when $x = \frac{5\pi}{12}$.

Give your answer correct to three significant figures. [3]

(b) Show that $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}\ln(\cot x)\right) = -\,\mathrm{cosec}\,2x$. [4]

(c) Show that $\sqrt{\cot x}$ is an integrating factor for this differential equation. [4]

(d) Hence, by solving the differential equation, show that $y = x\sqrt{\tan x}$. [5]

(e) Consider the curve $y = x\sqrt{\tan x}$ for $0 < x < \frac{\pi}{2}$ and the Euler's method approximation calculated in part (a).

(e.i) Find the y-coordinate at $x = \frac{5\pi}{12}$. Give your answer correct to three significant figures. [1]

(e.ii) By considering the gradient of the curve, suggest a reason why Euler's method does not give a good approximation for the $y$-coordinate at $x = \frac{5\pi}{12}$. [1]

(e.iii) State why this approximation is less than the $y$-coordinate at $x = \frac{5\pi}{12}$. [1]

(f) By considering $\frac{\mathrm{d}y}{\mathrm{d}x} - y\,\mathrm{cosec}\,2x = \sqrt{\tan x}$, deduce that the curve $y = x\sqrt{\tan x}$ has a positive gradient for $0 < x < \frac{\pi}{2}$. [2]