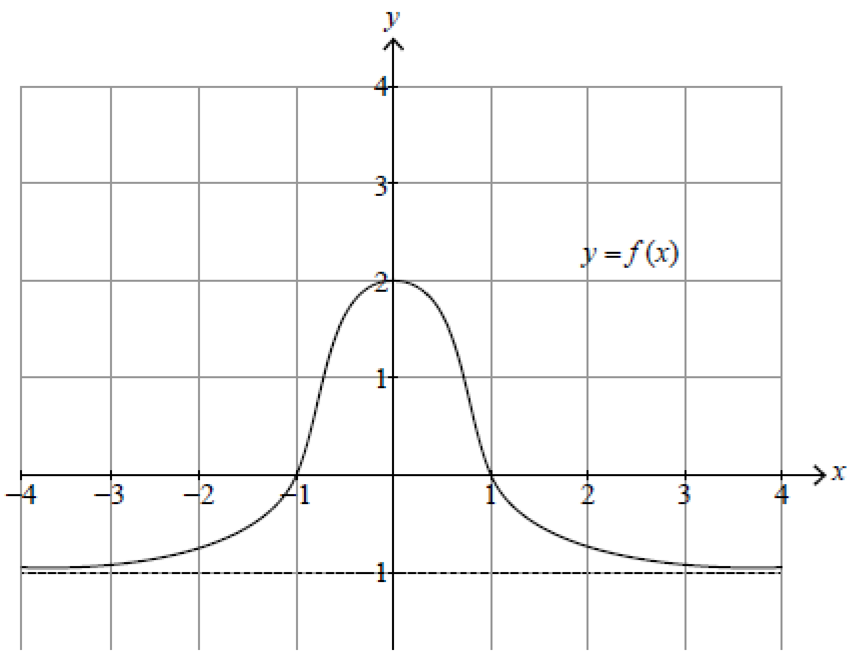
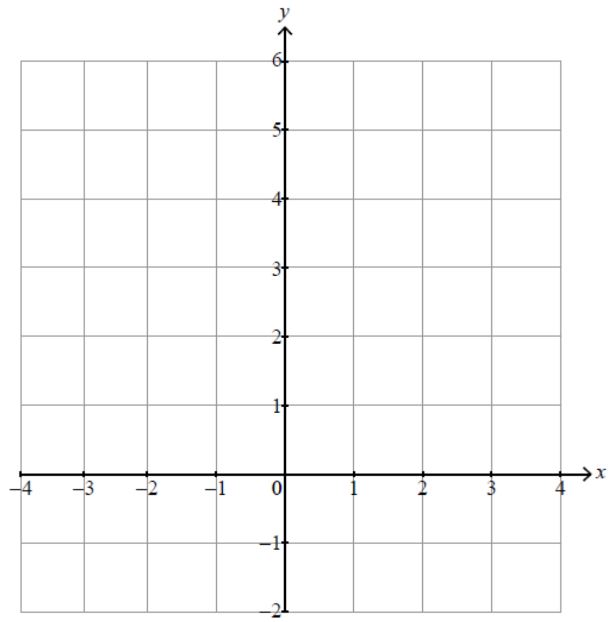
AAHL_Paper_1_QP [110 marks]

1. [Maximum mark: 5]

The following diagram shows the graph of $y = f(x)$. The graph has a horizontal asymptote at $y = -1$. The graph crosses the x-axis at $x = -1$ and $x = 1$, and the y-axis at $y = 2$.



On the following set of axes, sketch the graph of $y = [f(x)]^2 + 1$, clearly showing any asymptotes with their equations and the coordinates of any local maxima or minima.



[5]

2. [Maximum mark: 7]

A continuous random variable X has the probability density function f given by

$$f\left(x\right)=\begin{cases}\frac{\pi x}{36}\sin\left(\frac{\pi x}{6}\right), & 0\leqslant x\leqslant 6\\ 0, & \text{otherwise}\end{cases}.$$

Find $\mathrm{P}(0 \le X \le 3)$. [7]

3. [Maximum mark: 7]

The plane Π has the Cartesian equation $2x+y+2z=3$

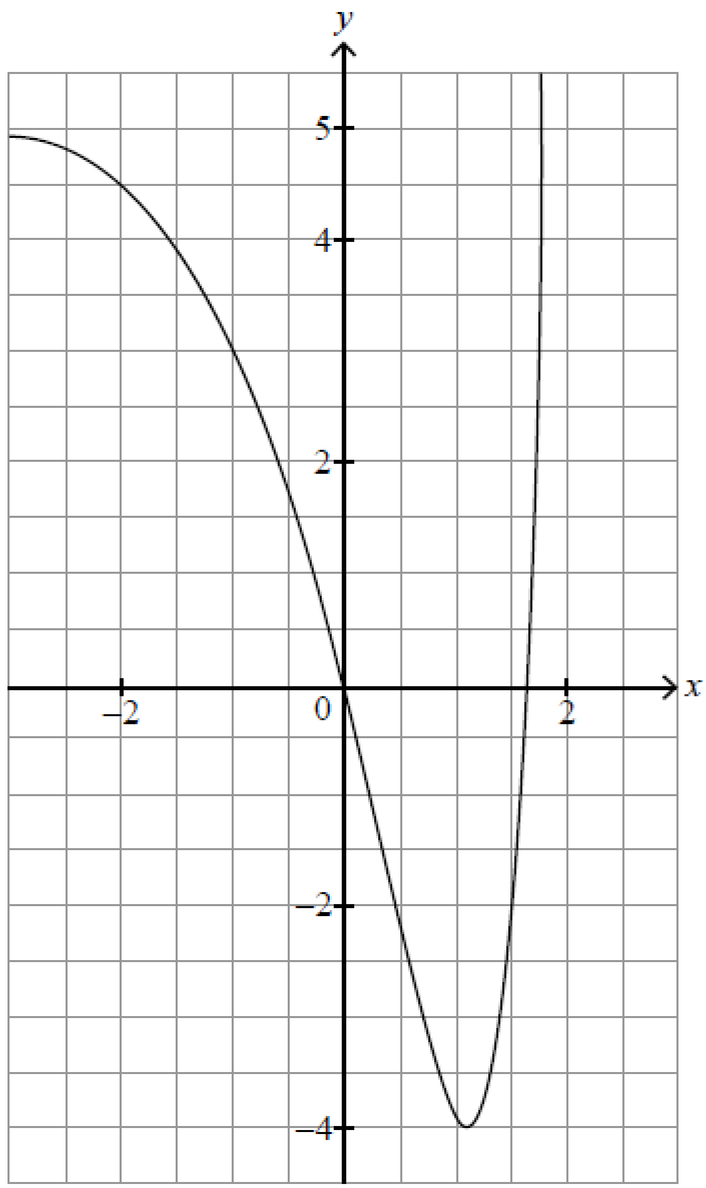
The line *L* has the vector equation ***r***

$$= \begin{pmatrix} 3 \\ -5 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ p \end{pmatrix},\ \mu, p \in \mathbb{R}.$$ The acute angle between the line *L* and the plane *Π* is 30°. [7]

Find the possible values of p.

4. [Maximum mark: 8]

The function f is defined by $f\left(x\right) = \mathrm{e}^{2x} - 6\mathrm{e}^{x} + 5,\ x \in \mathbb{R},\ x \leqslant a$. The graph of $y = f\left(x\right)$ is shown in the following diagram.

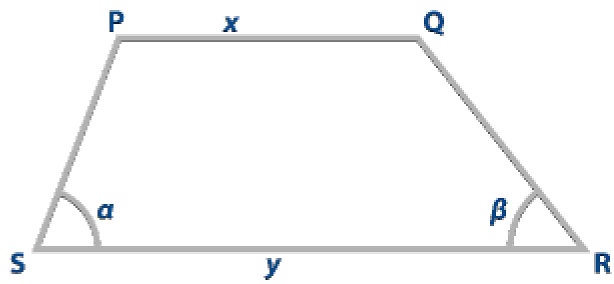


(a) Find the largest value of a such that f has an inverse function. [3]

(b) For this value of a, find an expression for $f^{-1}(x)$, stating its domain. [5]

5. [Maximum mark: 5]

Consider quadrilateral PQRS where $[\mathrm{PQ}]$ is parallel to $[\mathrm{SR}]$.



In PQRS, $\mathrm{PQ} = x$, $\mathrm{SR} = y$, $\mathrm{R\hat{S}P} = \alpha$ and $\mathrm{Q\hat{R}S} = \beta$. [5]

Find an expression for PS in terms of $x,\ y,\ \sin\beta$ and $\sin(\alpha+\beta)$.

6. [Maximum mark: 5]

The lines l_1 and l_2 have the following vector equations where $\lambda,\ \mu \in \mathbb{R}$ and $m \in \mathbb{R}$.

$$l_1 : \boldsymbol{r}_1 = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ m \end{pmatrix} \quad l_2 : \boldsymbol{r}_2 = \begin{pmatrix} -1 \\ -4 \\ -2m \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -5 \\ -m \end{pmatrix}$$

(a) Show that l_1 and l_2 are never perpendicular to each other. [3]

The plane Π has Cartesian equation $x + 4y - z = p$ where $p \in \mathbb{R}$.

Given that l_1 and Π have no points in common, find

(b) the value of m. [2]

7. [Maximum mark: 8]

Let $f(x) = \frac{2x+6}{x^2+6x+10},\ x \in \mathbb{R}$.

(a) Show that $f(x)$ has no vertical asymptotes. [3]

(b) Find the equation of the horizontal asymptote. [2]

(c) Find the exact value of $\int_0^1 f(x)\,dx$, giving the answer in the form $\ln q,\ q \in \mathbb{Q}$. [3]

8. [Maximum mark: 9]

Let $f(x) = \frac{x^2 - 10x + 5}{x + 1},\ x \in \mathbb{R},\ x \neq -1.$

(a) Find the co-ordinates of all stationary points. [4]

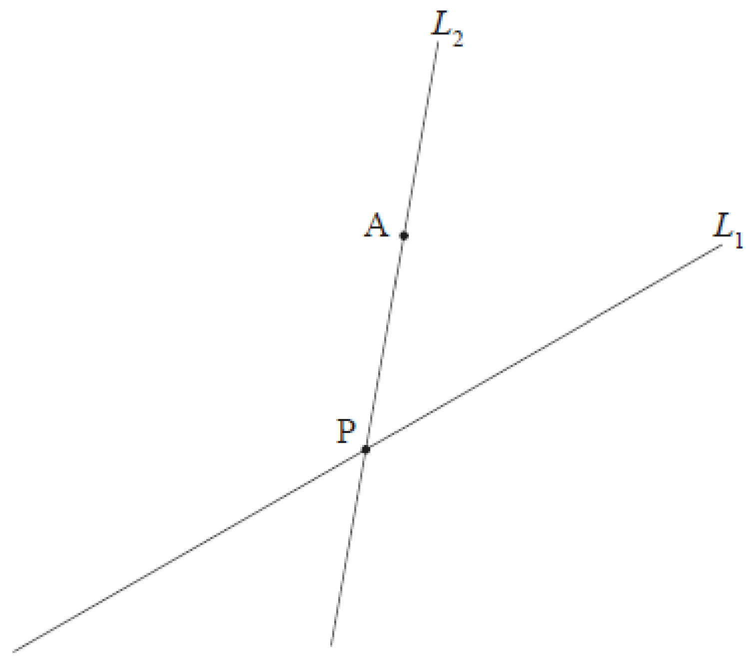
(b) Write down the equation of the vertical asymptote. [1]

(c) With justification, state if each stationary point is a minimum, maximum or horizontal point of inflection. [4]

9. [Maximum mark: 21]

Two lines, L_1 and L_2, intersect at point P. Point $\mathrm{A}(2t, 8, 3)$, where $t > 0$, lies on L_2. This is shown in the following diagram.

not to scale



The acute angle between the two lines is $\frac{\pi}{3}$.

The direction vector of L_1 is $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, and $\overrightarrow{\mathrm{PA}} = \begin{pmatrix} 2t \\ 0 \\ 3+t \end{pmatrix}$.

(a) Show that $4t = \sqrt{10t^2 + 12t + 18}$.

[4]

(b) Find the value of t. [4]

(c) Hence or otherwise, find the shortest distance from A to L_1.

[4]

A plane, Π, contains L_1 and L_2.

(d) Find a normal vector to Π. [2]

The base of a right cone lies in Π centred at A such that L_1 is a tangent to its base. The volume of the cone is $90\pi\sqrt{3}$ cubic units.

(e) Find the two possible positions of the vertex of the cone. [7]

10. [Maximum mark: 18]

Consider the series $\ln x + p \ln x + \frac{1}{3} \ln x + \ldots$, where $x \in \mathbb{R},\ x > 1$ and $p \in \mathbb{R},\ p \neq 0$.

Consider the case where the series is geometric.

(a.i) Show that $p = \pm \frac{1}{\sqrt{3}}$. [2]

(a.ii) Hence or otherwise, show that the series is convergent. [1]

(a.iii) Given that $p > 0$ and $S_\infty = 3 + \sqrt{3}$, find the value of x. [3]

Now consider the case where the series is arithmetic with common difference d.

(b.i) Show that $p = \frac{2}{3}$. [3]

(b.ii) Write down d in the form $k \ln x$, where $k \in \mathbb{Q}$. [1]

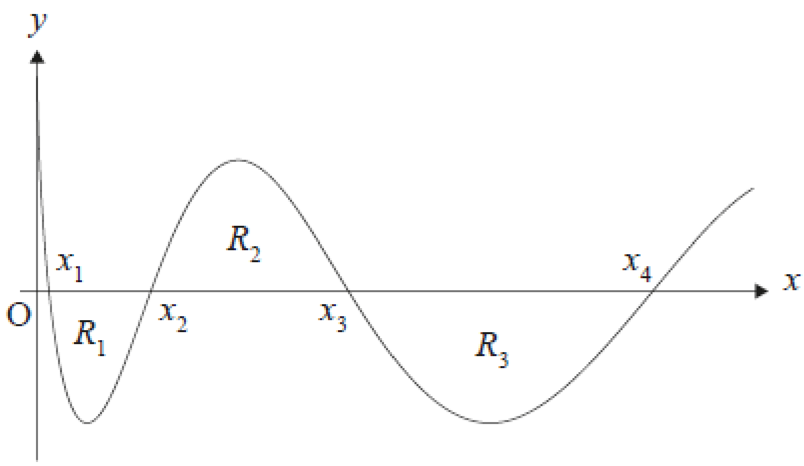
(b.iii) The sum of the first n terms of the series is $\ln\left(\frac{1}{x^3}\right)$.

Find the value of n. [8]

11. [Maximum mark: 17]

(a) By using an appropriate substitution, show that
$\int \cos \sqrt{x} \mathrm{d}x = 2\sqrt{x} \sin \sqrt{x} + 2 \cos \sqrt{x} + C$. [6]

The following diagram shows part of the curve $y = \cos \sqrt{x}$ for $x \geq 0$.



The curve intersects the x-axis at x_1, x_2, x_3, x_4 ….

The nth x-intercept of the curve, x_n, is given by $x_n = \frac{(2n-1)^2\pi^2}{4}$, where $n \in \mathbb{Z}^+$.

(b) Write down a similar expression for x_{n+1}. [1]

The regions bounded by the curve and the x-axis are denoted by $R_1, R_2, R_3, \ldots$, as shown on the above diagram.

(c) Calculate the area of region R_n.

Give your answer in the form $kn\pi$, where $k \in \mathbb{Z}^+$. [7]

(d) Hence, show that the areas of the regions bounded by the curve and the x-axis, $R_1, R_2, R_3, \ldots$, form an arithmetic sequence. [3]